

Recovering the stationary phase condition for accurately obtaining scattering and tunneling times

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(Dated: February 2, 2008)

Abstract

The stationary phase method is often employed for computing tunneling *phase* times of analytically-continuous *gaussian* or infinite-bandwidth step pulses which collide with a potential barrier. The indiscriminate utilization of this method without considering the barrier boundary effects leads to some misconceptions in the interpretation of the phase times. After reexamining the above barrier diffusion problem where we notice the wave packet collision necessarily leads to the possibility of multiple reflected and transmitted wave packets, we study the phase times for tunneling/reflecting particles in a framework where an idea of multiple wave packet decomposition is recovered. To partially overcome the analytical incongruities which rise up when tunneling phase time expressions are obtained, we present a theoretical exercise involving a symmetrical collision between two identical wave packets and a one dimensional squared potential barrier where the scattered wave packets can be recomposed by summing the amplitudes of simultaneously reflected and transmitted waves.

PACS numbers: 02.30.Mv, 03.65.Xp

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I. INTRODUCTION

The analytical methods utilized for reproducing the wave packet collision with a potential barrier have been widely discussed in the context of (one dimensional) scattering and tunneling phenomena [1, 2, 3]. In this extensively explored scenario, the stationary phase method (SPM) is the simplest and the most common approximation employed for obtaining the group velocity of a wave packet in a quantum scattering process which represents the collision of a particle with a square (rectangular) barrier of potential. First introduced to physics by Stokes and Kelvin [4], the SPM essentially provides an approximate value for the maximum of an integral in an analytical procedure which enables us to understand several subtleties of interesting quantum phenomena, such as tunneling [1, 5, 6], resonances [7], incidence-reflection and incidence-transmission interferences [2] as well as the Hartman effect [8] and the correspondent *superluminal* transition time interpretation [9, 10, 11]. It is also often quoted in the context of testing different theories for temporal quantities such as arrival, dwell and delay times [1, 3, 12] and the asymptotic behavior at long times [11, 13].

The study presented in this manuscript concerns with some limitations on the use of the SPM for obtaining the right position of a propagating wave packet subject to a very simple and common potential configuration. The limitations are not concerned with the domain of the method, i. e. with the mathematical basis and the analytical applicability of the method, they focus on the indiscriminate application of the method to time-dependent scattering problems and, in particular, for deriving phase times. The time-dependent scattering of a wave-packet by a potential barrier in the linear Schroedinger equation is interesting because it reconciles two types of wave theories. Those interested in linear theory often look at stationary (frequency-domain) solutions of a single frequency, whereas those interested in nonlinear theory usually look at time-dependent pulses. To implement the accurate analysis with the SPM, we work with the hypothesis of adding up the Fourier modes from the stationary solution to observe the evolution of a linear wave packet on collision with an obstacle. In the first part of the manuscript, the analysis of the applicability of the SPM is concentrated on the investigation of the non-relativistic diffusion of an incoming single wave packet with energy spectrum totally localized above the potential barrier. We are particularly interested in obtaining an analytical description of the above barrier diffusion problem by pointing out some interpretive problems which concern with non-conservation of probability during

the collision process. In the second part of our analysis, the correct quantification of the analytical incongruities which restrict the applicability of the principle of stationary phase is extended to the study a tunneling particle where, eventually, the *superluminal* interpretation of transition times can be ruined. In order to organize the ideas which are discussed in the manuscript, we have stated the following division. In section II we discuss the general applicability of the SPM in a wave packet propagation problem by introducing an analytical description of the above barrier diffusion problem where we reobtain the solution for the apparent physical inconsistency which concerns with the non-conservation of probability during the collision process. By recurring to a wave packet multiple peak decomposition [14], we can decompose the original colliding wave packet into *multiple* reflected and transmitted components by means of which the conservation of probabilities is recovered during the scattering process. The analytic expressions for all the reflected, transmitted and intermediary components are obtained and the validity of them is discussed in terms of an illustrative example where the analytic approximation and the reflects exactly the numerical results. The physical framework where the study of a tunneling process is embedded is permeated by theoretical constructions presented in section III. It involves analytically-continuous *gaussian* pulses which due to its not well-defined front lead to ambiguities in the interpretation of the wave packet speed of propagation. Beside it, infinite bandwidth signals cannot propagate through any real physical medium (whose transfer function is therefore finite) without pulse distortion, which also leads to ambiguities in determining propagation velocity during the tunneling process. Moreover, some of the barrier transposing time definitions lead, in tunneling time conditions, to very short times which can even become negative where they may seem to contradict simple concepts of causality. To partially overcome these incompatibilities, in section IV, we discuss a theoretical exercise involving a symmetrical collision with a one dimensional square potential where we recompose the scattered wave packet by summing the amplitudes of the reflected and transmitted waves in the scope of what we classify as a multiple peak decomposition analysis [14] where the conditions for the SPM applicability are totally recovered. Finally, we draw our conclusions in section V.

II. THE ABOVE BARRIER DIFFUSION PROBLEM

The principle of stationary phase to describe the group velocity is well-established, and was employed by Sommerfeld and Brillouin in their early studies of wave propagation and group velocity [15]. The method has in time become a standard tool for several theoretical applications [16] adopted not only by physicists but also by biologists, economists etc. It can also be used in a statistical sense, such that the most likely events tend to be associated with slowly-varying phase variation in the frequency domain, and unlikely events tend to be associated with rapidly varying phase with frequency.

To illustrate the applicability of the SPM into a physical problem it should be instructive to assume an equivalence between the complex variable z and a real one k so that an integral which represents a free quantum particle propagation can be identified by the wave packet solution of the unidimensional Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x, t) \quad (1)$$

with $V(x) = 0$, described in terms of the integral,

$$\psi(x, t) = \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} G(k, k_0) \exp [-i(E(k)t + kx - kx_0)], \quad (2)$$

where we have set $\hbar = 1$ and the dispersion relation as $E(k) = k^2/(2m)$. The integral (2) can therefore be estimated by finding the place where the phase has a vanishing derivative, evaluating (approximately) the integral in the neighborhood of this point. The movement of the peak coordinate of the wave packet $\psi(x, t)$ can be obtained by imposing the stationary phase condition

$$\frac{d}{dk} [E t - k(x - x_0)] \Big|_{k=k_0} = 0 \quad \Rightarrow \quad x_{\max} = x_0 + \frac{k_0}{m} t, \quad (3)$$

which means that the maximum of the wave packet propagates with a velocity $v = \frac{k_0}{m}$. In fact, we ratify this result when we explicitly calculate the integral of Eq. (2) by introducing a *gaussian* momentum distribution given by

$$G(k, k_0) = g(k - k_0) = \left(\frac{a^2}{2\pi} \right)^{\frac{1}{4}} \exp \left[-\frac{a^2(k - k_0)^2}{4} \right]. \quad (4)$$

In this case, the result of the integration (2) give us

$$\psi(x, t) = \varphi[x - x_0, t], \quad (5)$$

where

$$\varphi[x, t] = \left[\frac{\pi a^2}{2} \left(1 + \frac{4t^2}{m^2 a^4} \right) \right]^{-\frac{1}{4}} \exp \left[-\frac{\left(x - \frac{k_0}{m} t \right)^2}{a^2 \left(1 + \frac{2it}{ma^2} \right)} - \frac{i}{2} \arctan \left(\frac{2t}{ma^2} \right) + i(k_0 x - E_0 t) \right] \quad (6)$$

which evidently ratifies the result of Eq. (3).

Meanwhile, the method leads to some new interpretive discussion when the momentum distribution becomes a complex function, i. e. when $G(k, k_0)$ has to be written as $|G(k, k_0)| \exp[i\lambda(k)]$. In this case, the stationary phase condition for a free wave packet represented by Eq. (3) will be modified by the presence of an additional phase $\lambda(k)$. By assuming that $\lambda(k)$ can be expanded in the neighborhood of k_0 , i. e.

$$\lambda(k) \approx \lambda(k_0) + (k - k_0) \left. \frac{d\lambda(k)}{dk} \right|_{k=k_0} \quad (7)$$

the new stationary phase condition should give

$$x_{\max} = x_0 - \left. \frac{d\lambda(k)}{dk} \right|_{k=k_0} + \frac{k_0}{m} t. \quad (8)$$

As it is commonly presented in several textbooks of quantum mechanics [17], one can be persuaded to take the stationary phase condition as a necessary and *sufficient* statement for applying the SPM. Nevertheless, although the principle of stationary phase can be used to predict the most likely outcome of an experiment, it does not exclude the possibility of alternative outcomes. In fact, the results of the SPM depend critically upon the manipulation of the generic amplitude $G(k, k_0)$ prior to the application of the method. To clear up this point, let us firstly analyze the diffusion of a wave packet by a squared potential V_0 when the incoming energies are higher than V_0 . The stationary wave solution of the Schroedinger equation (1) computed by considering the potential barrier described by

$$\begin{aligned} 0 & \quad \text{if } x < 0 & \text{region I,} \\ V_0 & \quad \text{if } 0 < x < L & \text{region II,} \\ 0 & \quad \text{if } x > L & \text{region III,} \end{aligned} \quad (9)$$

can be decomposed into different wave functions for each interval of x , i. e.

$$\Phi(k, x) = \underbrace{\phi_{Inc}(k, x) + \phi_R(k, x)}_{\phi_I(k, x)} + \underbrace{\phi_\alpha(k, x) + \phi_\beta(k, x)}_{\phi_{II}(k, x)} + \underbrace{\phi_T(k, x)}_{\phi_{III}(k, x)}, \quad (10)$$

where

$$\begin{aligned}
\phi_{Inc}(k, x) &= \exp[i k x], \\
\phi_R(k, x) &= R(k) \exp[-i k x], \\
\phi_\alpha(k, x) &= \alpha(k) \exp[i q x], \\
\phi_\beta(k, x) &= \beta(k) \exp[-i q x], \\
\phi_T(k, x) &= T(k) \exp[i k x], \tag{11}
\end{aligned}$$

with $q = (k^2 - w^2)^{\frac{1}{2}}$ and $w = \sqrt{2mV_0}$. By solving the constraint equations at $x = 0$ and $x = L$ we obtain

$$\begin{aligned}
\alpha(k) &= \left[\frac{k(k+q)}{\mathcal{F}(k)} \right] \exp[i \Theta(k) - i q L], \\
\beta(k) &= - \left[\frac{k(k-q)}{\mathcal{F}(k)} \right] \exp[i \Theta(k) + i q L], \\
R(k) &= -i \left[\frac{k^2 - q^2}{\mathcal{F}(k)} \right] \sin[q L] \exp[i \Theta(k)], \\
T(k) &= \left[\frac{2 k q}{\mathcal{F}(k)} \right] \exp[i \Theta(k) - i k L], \tag{12}
\end{aligned}$$

where

$$\mathcal{F}(k) = \{4 k^2 q^2 \cos^2[q L] + (k^2 + q^2)^2 \sin^2[q L]\}^{\frac{1}{2}} \tag{13}$$

and

$$\Theta(k) = \arctan \left\{ \frac{k^2 + q^2}{2 k q} \tan[q L] \right\}. \tag{14}$$

The explicit expression for the correspondent propagating wave packets can be obtained by solving the integrals like

$$\psi_f(x, t) = \int_w^{+\infty} \frac{dk}{\sqrt{2\pi}} g(k - k_0) \phi_f(k, x) \exp[-i E t], \tag{15}$$

with $f \equiv \alpha, \beta, R, T$. As first approximation which is commonly used in quantum mechanics textbooks [17], we obtain the analytical formulas to these integrals by assuming the momentum distribution $g(k - k_0)$ is sufficiently sharply peaked around a maximum point $k_0 > w$. In this case, the integration can be extended from $[w, \infty]$ to $[-\infty, \infty]$ without modifying the final result. However, in the sense we are investigating, as we shall demonstrate in the following, it is *erroneously* assumed that the k -dependent phase terms $\Theta(k)$ and q can be

approximately described by a series expansion around $k = k_0$ up to the first order term, i.e.

$$\Theta(k) \approx \Theta(k_0) + (k - k_0)\Theta'(k_0) \quad \text{and} \quad q \approx q_0 + (k - k_0) \left. \frac{dq}{dk} \right|_{k=k_0} \quad (16)$$

where

$$\Theta'(k) = \frac{2m}{q} \left[\frac{(k^2 + q^2)k^2 q L - (k^2 - q^2)^2 \sin[qL] \cos[qL]}{4k^2 q^2 + (k^2 - q^2)^2 \sin^2[qL]} \right] \quad \text{and} \quad \left. \frac{dq}{dk} \right|_{k=k_0} = \frac{k_0}{q_0}.$$

At the same time, when k is approximated by k_0 , it is assumed that the modulating amplitude $|\phi_f(k, x)|$ can be put out of the integral giving the following results,

$$\begin{aligned} \psi_{Inc}(x, t) &\approx \varphi[x - x_0, t], \\ \psi_R(x, t) &\approx R(k_0) \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} g(k - k_0) \exp[-iE t - i k(x + x_0) + +i(k - k_0)\Theta'(k_0)] \\ &= R(k_0) \exp[-i k_0 \Theta'(k_0)] \varphi[-x - x_0 + \Theta'(k_0), t], \\ \psi_\alpha(x, t) &\approx \alpha(k_0) \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} g(k - k_0) \exp[-i(E t + k x_0 - q_0 x) + i(k - k_0) \left(\frac{k_0}{q_0}(x - L) + \Theta'(k_0) \right)] \\ &= \alpha(k_0) \exp \left[i q_0 x - i \frac{k_0^2}{q_0} (x - L) - i k_0 \Theta'(k_0) \right] \varphi \left[\frac{k_0}{q_0}(x - L) - x_0 + \Theta'(k_0), t \right], \\ \psi_\beta(x, t) &\approx \beta(k_0) \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} g(k - k_0) \exp[-i(E t + k x_0 + q_0 x) - i(k - k_0) \left(\frac{k_0}{q_0}(x - L) - \Theta'(k_0) \right)] \\ &= \beta(k_0) \exp \left[-i q_0 x + i \frac{k_0^2}{q_0} (x - L) - i k_0 \Theta'(k_0) \right] \varphi \left[-\frac{k_0}{q_0}(x - L) - x_0 + \Theta'(k_0), t \right], \\ \psi_T(x, t) &\approx T(k_0) \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}} g(k - k_0) \exp[-iE t + i k(x - x_0) + i(k - k_0)(\Theta'(k_0) - L)] \\ &= T(k_0) \exp[-i k_0 \Theta'(k_0)] \varphi[x - x_0 - L + \Theta'(k_0), t]. \end{aligned} \quad (17)$$

By observing the *gaussian* shape of $\varphi[x, t]$ given by Eq. (6), one can easily (but wrongly!) identify the position of the peak of the wave packets. The times corresponding to the position x of the incident and reflected wave packet peaks would be respectively given by

$$t_{Inc}(x) = \left[\frac{x - x_0}{v_k} \right] \Big|_{k=k_0} \quad \text{and} \quad t_R(x) = \left[-\frac{x + x_0 - \Theta'(k)}{v_k} \right] \Big|_{k=k_0} \quad (18)$$

where $v_k = \frac{dE}{dk} = \frac{k}{m}$. Only $x < 0$ is physical in this result since these waves, by definition, lie in region I. Since the phase of the incoming wave contains only the plane wave factors, i.e. it is devoid of $\Theta(k)$, the incoming peak reaches the barrier at $x = 0$ at time $t_{Inc}(0) = -(x_0/v_{k_0})$ (neglecting interference effects). The presence of the phase term $\Theta'(k_0) = (d\Theta(k)/dk)|_{k=k_0}$ for $t_R(x)$ would imply a time delay, or a time advance, which depends on the signal of $\Theta'(k_0)$,

for the reflected wave with respect to the incident one, i.e.

$$t_R(0) = t_{Inc}(0) - \frac{\Theta'(k)}{v_k} \Big|_{k=k_0} \quad (19)$$

It is analogous to what happens for the step potential tunneling penetration when $E < V_0$ [17]. When $0 < x < L$ in region II, the times for the “intermediary” α and β wave packet peaks would be given by

$$t_\alpha(x) = \left[\frac{(x-L)}{v_q} - \frac{(x_0 - \Theta'(k))}{v_k} \right] \Big|_{k=k_0} \quad \text{and} \quad t_\beta(x) = \left[-\frac{(x-L)}{v_q} - \frac{(x_0 - \Theta'(k))}{v_k} \right] \Big|_{k=k_0} \quad (20)$$

And finally, when $x > L$ in region III, the peak of the transmitted wave packet would be written as

$$t_T(x) = \left[\frac{x - x_0 - L + \Theta'(k)}{v_k} \right] \Big|_{k=k_0} \quad (21)$$

The above results (18-21) could be directly obtained by applying the SPM to the wave functions $\psi_f(x, t)$ expressed in Eq. (15), where the stationary phases for each decomposed wave component f would be given by

$$\begin{aligned} \vartheta_{Inc}(x, t, k) &= -E t + k(x - x_0), \\ \vartheta_R(x, t, k) &= -E t - k(x + x_0) + \Theta(k), \\ \vartheta_\alpha(x, t, k) &= -E t - k x_0 + q(x - L) + \Theta(k), \\ \vartheta_\beta(x, t, k) &= -E t - k x_0 - q(x - L) + \Theta(k), \\ \vartheta_T(x, t, k) &= -E t + k(x - x_0 - L) + \Theta(k). \end{aligned} \quad (22)$$

Meanwhile, these time-dependencies must be *carefully interpreted*. In the Fig.(2), we illustrate the wave packet diffusion described analytically by the results of Eq. (17). The “pictures” display the wave function in the *proximity* of the barrier for suitably chosen times. By taking separately the right (α) and left (β) moving components in region II, independent of the value of $\Theta(k)$, we can observe that

$$\Delta t_\alpha = t_\alpha(L) - t_\alpha(0) = \Delta t_\beta = t_\beta(0) - t_\beta(L) = \frac{L}{v_q} \Big|_{k=k_0} \quad (23)$$

which should corresponds to transit time values “classically” expected. However, by considering the information carried by the wave packet peaks, we would have a complete time discontinuity at $x = 0$ represented by Eq. (19) and by the fact that

$$t_\alpha(0) = - \left[\frac{x_0}{v_k} + \frac{L}{v_q} - \frac{\Theta'(k)}{v_k} \right] \Big|_{k=k_0} \neq t_{Inc}(0), \quad (24)$$

i. e. the α wave could appear in region II at a time $t_\alpha(0)$ before (or after) the incident wave having arrived at $x = 0$. Both results expressed by Eqs. (19) and (24) which are illustrated in Fig. 2 should imply the non-conservation of probabilities. It is, by principle, unacceptable.

To be more accurate with this analysis and clear up some dubious points, it should be more convenient to verify two simple particular cases. In order to simplify the calculations, we set $x_0 = 0$ and we choose $q_0 L = n\pi$ ($n = 1, 2, 3, \dots$) so that we can write,

$$\mathcal{F}(k_0) = 2k_0 q_0, \quad \Theta(k_0) = 0, \quad R(k_0) = 0, \quad T(k_0) = 1 \quad (25)$$

and

$$\Theta'(k) = \frac{m L}{q_0} \frac{k_0^2 + q_0^2}{2k_0 q_0} > \frac{m L}{q_0} \quad (26)$$

so that

$$t_\alpha(0) = \frac{m L}{q_0} \frac{(k_0 - q_0)^2}{2k_0 q_0} > 0. \quad (27)$$

Otherwise, if we choose $q_0 L = (n + \frac{1}{2})\pi$, we shall obtain,

$$\mathcal{F}(k_0) = k_0^2 + q_0^2, \quad \Theta(k) = \frac{\pi}{2}, \quad R(k_0) = \frac{k_0^2 - q_0^2}{k_0^2 + q_0^2}, \quad T(k_0) = \frac{2k_0 q_0}{k_0^2 + q_0^2} \exp \left[i \left(\frac{\pi}{2} - k_0 L \right) \right] \quad (28)$$

and

$$\Theta'(k) = \frac{m L}{q_0} \frac{2k_0 q_0}{k_0^2 + q_0^2} < \frac{m L}{q_0} \quad (29)$$

which gives

$$t_\alpha(0) = -\frac{m L}{q_0} \frac{(k_0 - q_0)^2}{k_0^2 + q_0^2} < 0. \quad (30)$$

The latter *negative* value corroborates with the absurd possibility of the peak associated with the α wave appears before the peak of the incident wave packet arrival at $x = 0$. In fact, we must observe that if we proceed with a more general analysis, the phase derivative $\Theta'(k)$ does not present a regular behavior. It oscillates very rapidly, as we can observe in Fig.(1) so that the SPM *cannot* be applied!

Other incongruities in the naive application of the SPM presented above have been pointed out [14]. Differently from the analytical analysis presented above, the numerical simulations of a wave packet diffusion above the potential barrier shows the appearance of multiple peaks due to the two reflection points at $x = 0$ and $x = L$ (see the figures). In fact, numerical calculations automatically conserve probabilities, at least to within the numerical errors. This observation suggest a new analysis and subsequent interpretation of the ambiguities presented in the previous section. In order to correctly apply the SPM and

accurately obtain the position of the peak of the propagating wave packet with an accurate time dependence, we are constrict to solve the continuity constraints of the Schroedinger equation at each potential discontinuity point $x = 0$ and $x = L$ by considering multiple successive reflections and transmissions. The phenomenon was already described in [14] and consists in an incoming wave of unitary amplitude with momentum distribution centered at k_0 which reaches the interface $x = 0$ where, at an instant $t = t_0$, is decomposed into a reflected wave component of amplitude (see the diagram) R_1 and a transmitted wave component of amplitude α_1 . The transmitted wave continues propagating until it reaches $x = L$ where, at $t = t_1$, it is now decomposed into a reflected wave component of amplitude β_1 and a transmitted wave component of amplitude T_1 . That new reflected wave component comes back to $x = 0$ where, at $t = t_2$, it is again decomposed into another reflected wave component of amplitude α_2 and transmitted wave component of amplitude R_2 which will be added to R_1 . This iterative process continues for infinity times as we can see in the following diagram,

$$\begin{array}{c}
 \left. \begin{array}{c} \exp[i k x] + R_1 \exp[-i k x] \\ R_2 \exp[-i k x] \\ \vdots \\ R_n \exp[-i k x] \end{array} \right| \begin{array}{c} \alpha_1 \exp[i q x] \\ \alpha_2 \exp[i q x] + \beta_1 \exp[-i q x] \\ \vdots \\ \alpha_n \exp[i q x] + \beta_{n-1} \exp[-i q x] \end{array} \left| \begin{array}{c} \alpha_1 \exp[i q x] + \beta_1 \exp[-i q x] \\ \alpha_2 \exp[i q x] + \beta_2 \exp[-i q x] \\ \vdots \\ \alpha_n \exp[i q x] + \beta_n \exp[-i q x] \end{array} \right| \begin{array}{c} T_1 \exp[i k x] \\ T_2 \exp[i k x] \\ \vdots \\ T_n \exp[i k x] \end{array} \\
 x = 0 \qquad \qquad \qquad x = L
 \end{array}$$

The continuity constraints over $\psi(x, t)$ for each potential step at $x = 0$ and $x = L$ determine the coefficients

$$\begin{aligned}
 R_1 &= \frac{k - q}{k + q}, \quad \alpha_1 = \frac{2k}{k + q}, \quad \beta_1 = \frac{2k(q - k)}{(k + q)^2} \exp[2i q L], \\
 T_1 &= \frac{4kq}{(k + q)^2} \exp[i(q - k)L], \quad R_2 = \frac{q}{k} \alpha_1 \beta_1,
 \end{aligned} \tag{31}$$

and establish the recurrence relations

$$\frac{R_{n+2}}{R_{n+1}} = \frac{\alpha_{n+1}}{\alpha_n} = \frac{\beta_{n+1}}{\beta_n} = \frac{T_{n+1}}{T_n} = \left(\frac{k - q}{k + q} \right)^2 \exp[2i q L] \quad n = 1, 2, \dots, \tag{32}$$

which allow us to write the sum of the coefficients R_n , α_n , β_n , and T_n as

$$\begin{aligned} R &= \sum_{n=1}^{\infty} R_n = R_1 + R_2 \left[1 - \left(\frac{k-q}{k+q} \right)^2 \exp[2i q L] \right]^{-1}, \\ \alpha &= \sum_{n=1}^{\infty} \alpha_n = \alpha_1 \left[1 - \left(\frac{k-q}{k+q} \right)^2 \exp[2i q L] \right]^{-1}, \\ \beta &= \sum_{n=1}^{\infty} \beta_n = \beta_1 \left[1 - \left(\frac{k-q}{k+q} \right)^2 \exp[2i q L] \right]^{-1}, \\ T &= \sum_{n=1}^{\infty} T_n = T_1 \left[1 - \left(\frac{k-q}{k+q} \right)^2 \exp[2i q L] \right]^{-1}. \end{aligned} \quad (33)$$

These sums reproduce exactly the expressions in Eq. (12). In this form the interpretation is easy. R_1 represents the first reflected wave (it has no time delay since it is real). R_2 represents the second reflected wave and, as a consequence of the continuity condition at $x = 0$, it is the sum, in region II, of the first left-going wave (β_1) and the second right-going amplitude (α_2), i.e.

$$R_2 = \alpha_2 + \beta_1 \equiv \frac{q}{k} \alpha_1 \beta_1.$$

This structure is that given by considering two “step functions” back-to-back. Thus at each interface the “reflected” and “transmitted” waves are instantaneous i. e. without any delay time.

Now we can calculate the time at which each transmitted or reflected wave appears by applying the SPM for each component of the total transmitted T or reflected R coefficients. It will give us the recurrence relation

$$t_n = t_{n-1} + \frac{m(x-L)}{q_0} \quad (34)$$

which coincides with the “classically” predicted value for the velocity of the particle above the barrier. Indeed the SPM *applied separately* to each term in the above series expansion for R yields delay multiples of $2 (dq/dE)|_{q=q_0} L = 2(m/q_0) L$. This agrees perfectly with the fact that since the peak momentum in region II is q_0 , the α and β waves have group velocities of q_0/m and hence transit times (one way) of $(m/q_0) L$. The first transmitted peak appears (according to this version of the SPM) after a time $(m/q_0) L$, in perfect agreement with the above interpretation.

Is this compatible with probability conservation? It is because of the following identity

$$\sum_{n=1}^{\infty} (|R_n|^2 + |T_n|^2) = 1. \quad (35)$$

This result is by no means obvious since it coexists with the well-known result, from the plane wave analysis,

$$|R|^2 + |T|^2 = \left| \sum_{n=1}^{\infty} R_n \right|^2 + \left| \sum_{n=1}^{\infty} T_n \right|^2 = 1. \quad (36)$$

To conclude, we can state that the conditions for applying the SPM depend upon the correct manipulation of the amplitude. *A posteriori* this seems obvious but the point is that unless we know the number of separate peaks the SPM is ambiguous. There is, however, a converse to this question. If the modulating function is such that two or more wave packets overlap, then we cannot treat them separately without considering all the interference effects. The above barrier analysis is simply a particular example of this ambiguity.

In order to obtain an analytic formula for the wave packet propagation, we consider the *gaussian* momentum distribution given by Eq. (4). Using the linear approximation for the momentum q given by (16) and considering the expressions in (33) resulting from the multiple peak decomposition in (15), again, with the same pertinent approximations used for obtaining (17), we can analytically construct the following simplified expressions

$$\begin{aligned} \psi_I(x, t) &= \varphi[x - x_0, t], \\ \psi_R(x, t) &= \frac{k_0 - q_0}{k_0 + q_0} \varphi[-x - x_0, t] + \\ &\quad \frac{4k_0 q_0 (q_0 - k_0)}{(k_0 + q_0)^3} \exp \left[-i \frac{w^2}{q_0} L \right] \sum_{n=0}^{\infty} \left(\frac{k_0 - q_0}{k_0 + q_0} \exp \left[-i \frac{w^2}{q_0} L \right] \right)^{2n} \varphi[-x - x_0 + 2(n+1) \frac{k_0}{q_0} L, t], \\ \psi_{\alpha}(x, t) &= \frac{2k_0}{k_0 + q_0} \exp \left[-i \frac{w^2}{q_0} x \right] \sum_{n=0}^{\infty} \left(\frac{k_0 - q_0}{k_0 + q_0} \exp \left[-i \frac{w^2}{q_0} L \right] \right)^{2n} \varphi[(x + 2nL) \frac{k_0}{q_0} - x_0, t], \\ \psi_{\beta}(x, t) &= \frac{2k_0 (q_0 - k_0)}{(k_0 + q_0)^2} \exp \left[i \frac{w^2}{q_0} (x - 2L) \right] \sum_{n=0}^{\infty} \left(\frac{k_0 - q_0}{k_0 + q_0} \exp \left[-i \frac{w^2}{q_0} L \right] \right)^{2n} \varphi[(2nL + 2L - x) \frac{k_0}{q_0} - x_0, t], \\ \psi_T(x, t) &= \frac{4k_0 q_0}{(k_0 + q_0)^2} \exp \left[-i \frac{w^2}{q_0} L \right] \sum_{n=0}^{\infty} \left(\frac{k_0 - q_0}{k_0 + q_0} \exp \left[-i \frac{w^2}{q_0} L \right] \right)^{2n} \varphi[x - x_0 - L + (2n+1) \frac{k_0}{q_0} L, t]. \end{aligned} \quad (37)$$

which allow us to visualize the multiple peak decomposition illustrated in Fig. 3. By comparing with the exact numerical results, the analytic approximations expressed in (37) will be valid only under certain restrictive conditions. To explain such a statement, when we analytically solve the integrals like (15), the exponential $\exp[2i q L]$ which appears in the recurrence relations (32) are approximated by

$$\exp[2i q_0 L + 2i(k - k_0) \frac{k_0}{q_0}]. \quad (38)$$

In this case, the above analytic expressions can be obtained only when the exponential function does not oscillate in the interval of relevance of the envelop *gaussian* function $g(k - k_0)$, i. e. when $\Delta q L < \pi$, which can be expressed in terms of the wave packet width as

$$\Delta q L \approx \frac{dq}{dk} \bigg|_{k=k_0} \quad \Delta k L \approx \frac{k_0 L}{q_0 a} < \pi. \quad (39)$$

where we have used $\Delta k \approx 1/a$. The above constraint implicitly carries a very peculiar character: the multiple peak decomposition can also be evidenced when the wave packet width a is larger than the potential barrier width L , i. e. when we assume the relation k_0/q_0 is in the interval $1 < k_0/q_0 < \pi$ we can have propagating wave packets with $a > L$ satisfying the requirements for the multiple peak resolution (37).

III. LIMITATIONS ON THE TUNNELING TIME ANALYSIS

On the theoretical front, people have tried to introduce quantities that have the dimension of time and can somehow be associated with the passage of the particle through the barrier or, strictly speaking, with the definition of the tunneling time. Since a long time these efforts have led to the introduction of several *time* definitions [6, 11, 18, 19, 20, 21, 22, 23, 24, 25], some of which are completely unrelated to the others, which can be organized into three groups: (1) The first group comprises a time-dependent description in terms of wave packets where some features of an incident wave packet and the comparable features of the transmitted packet are utilized to describe a *delay* as tunneling time [1]. (2) In the second group the tunneling times are computed with basis on averages over a set of kinematical paths, whose distribution is supposed to describe the particle motion inside a barrier, i. e. Feynman paths are used like real paths to calculate an average tunneling time with the weighting function $\exp [i S x(t)/\hbar]$, where S is the action associated with the path $x(t)$ (where $x(t)$ represents the Feynman paths initiated from a point on the left of the barrier and ending at another point on the right of it [26]). The Wigner distribution paths [6], and the Bohm approach [27, 28] are included in this group. (3) In the third group we notice the introduction of a new degree of freedom, constituting a physical clock for the measurements of tunneling times. Separately, it stands by itself the dwell time approach. The time related to the tunneling process is defined by the interval during which the incident flux has to exist and act, to provide the expected accumulated particle storage, inside the barrier [3]. The

methods with a Larmor clock [18] or an oscillating barrier [29] are comprised by this group.

There is no general agreement [9, 25] among the above definitions about the meaning of tunneling times (some of the proposed tunneling times are actually traversal times, while others seem to represent in reality only the spread of their distributions) and about which, if any, of them is the proper tunneling time, in particular, due to the following reasons [25]: (a) the problem of defining tunneling times is closely connected with the more general definition of the quantum-collision duration, and therefore with the fundamental fact that in quantum mechanics, time enters as a parameter rather than an observable to which an operator can be assigned (b) the motion of particles inside a potential barrier is a quantum phenomenon, that till now has been devoid of any direct classical limit; (c) there are essential differences among the initial, boundary and external conditions assumed within the various definitions proposed in the literature; those differences have not been sufficiently analyzed yet. In particular, the study of tunneling mechanisms is embedded by theoretical constructions involving analytically-continuous *gaussian*, or infinite-bandwidth step pulses to examine the tunneling process. Nevertheless, such holomorphic functions do not have a well-defined front in a manner that the interpretation of the wave packet speed of propagation becomes ambiguous. Moreover, infinite bandwidth signals cannot propagate through any real physical medium (whose transfer function is therefore finite) without pulse distortion, which also leads to ambiguities in determining the propagation velocity during the tunneling process. For instance, some of the barrier traversal time definitions lead, in tunneling time conditions, to very short times which can even become negative, precipitately inducing to an interpretation of violation of simple concepts of causality. Otherwise, negative speeds do not seem to create problems with causality, since they were predicted both within special relativity and within quantum mechanics [23]. A possible explanation of the time advancements related to the negative speeds can come, in any case, from consideration of the very rapid spreading of the initial and transmitted wave packets for large momentum distribution widths. Due to the similarities between tunneling (quantum) packets and evanescent (classical) waves, exactly the same phenomena are to be expected in the case of classical barriers (we can mention the analogy between the stationary Helmholtz equation for an electromagnetic wave packet - in a waveguide, for instance - in the presence of a *classical* barrier and the stationary Schroedinger equation, in the presence of a potential barrier [3, 11, 30]). The existence of such negative times is predicted by relativity itself, on the basis of the ordinary postulates

[25], and they appear to have been experimentally detected in many works [31, 32].

In this extensively explored scenario, the first group quoted above contains the so-called phase times [33, 34] which are obtained when the stationary phase method (SPM) is employed for obtaining the times related to the motion of the wave packet spatial centroid which adopts averages over fluxes pointing in a well-defined direction only, and has recourse to a quantum operator for time [25]. Generically speaking, the SPM essentially enables us to parameterize some subtleties of several quantum phenomena, such as tunneling [1, 5, 6], resonances [7], incidence-reflection and incidence-transmission interferences [2] as well as the Hartman Effect [8] and its *superluminal* traversal time interpretation [3, 11, 25]. In fact, it is the simplest and commonest approximation method for describing the group velocity of a wave packet in a quantum scattering process represented by a collision of a particle with a potential barrier [3, 8, 14, 19, 25, 34].

In the following, our attention is particularly concentrated on some limitations on the use of the SPM for deriving tunneling times for which we furnish an accurate quantification of the analytical incongruities which restrict the applicability of this method. We introduce a theoretical construction involving a symmetrical collision with a unidimensional square potential where the scattered wave packets can be reconstructed by summing the amplitudes of the reflected and transmitted waves in the scope of what we denominate a multiple peak decomposition analysis [14] in a manner that the analytical conditions for the SPM applicability are totally recovered.

Generically speaking, the SPM can be successfully utilized for describing the movement of the center of a wave packet constructed in terms of a momentum distribution $g(k - k_0)$ which has a pronounced peak around k_0 . By assuming the phase which characterizes the propagation varies sufficiently smoothly around the maximum of $g(k - k_0)$, the stationary phase condition enable us to calculate the position of the peak of the wave packet (highest probability region to find the propagating particle). With regard to the tunneling effect, the method is indiscriminately applied to find the position of a wave packet which traverse a potential barrier.

For the case we consider the potential barrier

$$V(x) = \begin{cases} V_0 & x \in [-L/2, L/2] \\ 0 & x \notin [-L/2, L/2] \end{cases} \quad (40)$$

it is well known that the transmitted wave packet solution ($x \geq L/2$) calculated by means of the Schroedinger formalism is given by [17]

$$\psi^T(x, t) = \int_0^w \frac{dk}{2\pi} g(k - k_0) |T(k, L)| \exp \left[i k (x - L/2) - i \frac{k^2}{2m} t + i \Theta(k, L) \right] \quad (41)$$

where, in case of tunneling, the transmitted amplitude is written as

$$|T(k, L)| = \left\{ 1 + \frac{w^4}{4k^2 \rho^2(k)} \sinh^2 [\rho(k) L] \right\}^{-\frac{1}{2}}, \quad (42)$$

and the phase shift is described in terms of

$$\Theta(k, L) = \arctan \left\{ \frac{2k^2 - w^2}{k \rho(k)} \tanh [\rho(k) L] \right\}, \quad (43)$$

for which we have made explicit the dependence on the barrier length L and we have adopted $\rho(k) = (w^2 - k^2)^{\frac{1}{2}}$ with $w = (2mV_0)^{\frac{1}{2}}$ and $\hbar = 1$. Without thinking over an eventual distortion that $|T(k, L)|$ causes to the supposedly symmetric function $g(k - k_0)$, when the stationary phase condition is applied to the phase of Eq. (42) we obtain

$$\begin{aligned} \frac{d}{dk} \left\{ k (x - L/2) - \frac{k^2}{2m} t + \Theta(k, L) \right\} \bigg|_{k=k_{\max}} &= 0 \quad \Rightarrow \\ x - L/2 - \frac{k_{\max}}{m} t + \frac{d\Theta(k, L)}{dk} \bigg|_{k=k_{\max}} &= 0. \end{aligned} \quad (44)$$

The above result is frequently adopted for calculating the transit time t_T of a transmitted wave packet when its peak emerges at $x = L/2$,

$$t_T = \frac{m}{k_{\max}} \frac{d\Theta(k, \alpha(L))}{dk} \bigg|_{k=k_{\max}} = \frac{2mL}{k_{\max} \alpha} \left\{ \frac{w^4 \sinh(\alpha) \cosh(\alpha) - (2k_{\max}^2 - w^2) k_{\max}^2 \alpha}{4k_{\max}^2 (w^2 - k_{\max}^2) + w^4 \sinh^2(\alpha)} \right\} \quad (45)$$

where we have defined the parameter $\alpha = (w^2 - k_{\max}^2)^{\frac{1}{2}} L$. The concept of *opaque* limit is introduced when we assume that k_{\max} is independent of L and then we make α tends to ∞ [11]. In this case, the transit time can be rewritten as

$$t_T^{OL} = \frac{2m}{k_{\max} \rho(k_{\max})}. \quad (46)$$

In the literature, the value of k_{\max} is frequently approximated by k_0 , the maximum of $g(k - k_0)$, which, in fact, does not depend on L and could lead us to the transmission time *superluminal*

interpretation [9, 11, 35] of (46). To clear up this point, we notice that when we take the *opaque* limit with L going to ∞ and w fixed as well as with w going to ∞ and L fixed, with $k_0 < w$ in both cases, the expression (46) suggests times corresponding to a transmission process performed with velocities higher than c [11]. Such a *superluminal* interpretation was extended to the study of quantum tunneling through two successive barriers separated by a free region where it was theoretically verified that the total traversal time does not depend not only on the barrier widths, but also on the free region extension [9]. Besides, in a subsequent analysis, the same technique was applied to a problem with multiple successive barrier where the tunneling process was designated as a highly non-local phenomenon [35].

It would be perfectly acceptable to consider $k_{\max} = k_0$ for the application of the stationary phase condition when the momentum distribution $g(k - k_0)$ centered at k_0 is not modified by any bound condition. This is the case of the incident wave packet before colliding with the potential barrier. Our criticism concern, however, with the way of obtaining all the above results for the transmitted wave packet. It does not have taken into account the bounds and enhancements imposed by the analytical form of the transmission coefficient.

Hartman himself had already noticed that the

“Discussion of the time dependent behavior of a wave packet is complicated by its diffuse or spreading nature; however, the position of the peak of a symmetrical packet can be described with some precision”

To perform the correct analysis, we should calculate the right value of k_{\max} to be substituted in Eq. (45) before taking the *opaque* limit. We are obliged to consider the relevant amplitude for the transmitted wave as the product of a symmetric momentum distribution $g(k - k_0)$ which describes the *incoming* wave packet by the modulus of the transmission amplitude $T(k, L)$ which is a crescent function of k . The maximum of this product representing the transmission modulating function would be given by the solution of the equation

$$g(k - k_0) |T(k, L)| \left[\frac{g'(k - k_0)}{g(k - k_0)} + \frac{|T(k, L)|'}{|T(k, L)|} \right] = 0 \quad (47)$$

Obviously, the peak of the modified momentum distribution is shifted to the right of k_0 so that k_{\max} have to be found in the interval $]k_0, w[$. Moreover, we confirm that k_{\max} presents an implicit dependence on L as we can demonstrate by the numerical results presented in *Table 1* where we have found the maximum of $g(k - k_0) |T(k, L)|$ by assuming $g(k - k_0)$ is

a *gaussian* function almost completely comprised in the interval $[0, w]$ given by

$$g(k - k_0) = \left(\frac{a^2}{2\pi} \right)^{\frac{1}{4}} \exp \left[-\frac{a^2(k - k_0)^2}{4} \right]. \quad (48)$$

By increasing the value of L with respect to a , the value of k_{\max} obtained from the numerical calculations to be substituted in Eq. (45) also increases until L reaches certain values for which the modified momentum distribution becomes unavoidably distorted. In this case, the relevant values of k are concentrated around the upper bound value w . We shall show in the following that the value of L which starts to distort the momentum distribution can be analytically obtained in terms of a .

Now, if we take the *opaque* limit of α by fixing L and increasing w , the above results immediately ruin the *superluminal* interpretation upon the result of Eq. (45) since t_T^{OL} tends to ∞ when k_{\max} is substituted by w .

Otherwise, when w is fixed and L tends to ∞ , the parameter α calculated at $k = w$ becomes indeterminate. The transit time t_T still tends to ∞ but now it exhibits a peculiar dependence on L which can be easily observed by defining the auxiliary function

$$G(\alpha) = \frac{\sinh(\alpha) \cosh(\alpha) - \alpha}{\sinh^2(\alpha)} \quad (49)$$

which allow us to write

$$t_T^\alpha = \frac{2mL}{w\alpha} G(\alpha). \quad (50)$$

When $\alpha \gg 1$, the transmission time always assume infinite values with an asymptotic dependence on $(w^2 - k^2)^{-\frac{1}{2}}$,

$$t_T^\alpha \approx \frac{2m}{w(w^2 - k^2)^{\frac{1}{2}}} \rightarrow \infty. \quad (51)$$

Only when α tends to 0 we have an explicit linear dependence on L given by

$$t_T^0 = \frac{2mL}{w} \lim_{\alpha \rightarrow 0} \left\{ \frac{G(\alpha)}{\alpha} \right\} = \frac{4mL}{3w} \quad (52)$$

In addition to the above results, the transmitted wave must be carefully studied in terms of the rapport between the barrier extension L and the wave packet width a . For very thin barriers, i. e. when L is much smaller than a , the modified transmitted wave packet presents substantially the same form of the incident one. For thicker barriers, but yet with $L < a$, the peak of the *gaussian* wave packet modulated by the transmission coefficient is shifted to higher energy values, i. e. $k_{\max} > k_0$ increases with L . For very thick barriers, i. e. when $L > a$, we are able to observe that the form of the transmitted wave packet is badly

distorted with the greatest contribution coming from the Fourier components corresponding to the energy w just above the top of the barrier in a kind of *filter effect*. We observe that the quoted distortion starts to appear when the modulated momentum distribution presents a *local maximal* point at $k = w$ which occurs when

$$\frac{d}{dk} [g(k - k_0) |T(k, L)|] \Big|_{k=w} > 0. \quad (53)$$

Since the derivative of the *gaussian* function $g(k - k_0)$ is negative at $k = w$, the Eq. (53) gives the relation

$$-\frac{g'(w - k_0)}{g(w - k_0)} < \lim_{k \rightarrow w} \left[\frac{T'(k, L)}{T(k, L)} \right] = \frac{w L^2}{4} \frac{\left(1 + \frac{w L^2}{3}\right)}{\left(1 + \frac{w L^2}{4}\right)} < \frac{w L^2}{3} \quad (54)$$

which effectively represents the inequality

$$\frac{a^2}{2} (w - k_0) < \frac{w L^2}{3} \Rightarrow L > \sqrt{\frac{3}{2}} a \left(1 - \frac{k_0}{w}\right). \quad (55)$$

Due to the *filter effect*, the amplitude of the transmitted wave is essentially composed by the plane wave components of the front tail of the *incoming* wave packet which reaches the first barrier interface before the peak arrival. Meanwhile, only whether we had *cut* the momentum distribution *off* at a value of k smaller than w , i. e. $k \approx (1 - \delta)w$, the *superluminal* interpretation of the transition time (46) could be recovered. In this case, independently of the way as α tends to ∞ , the value assumed by the transit time would be approximated by $t_T^\alpha \approx 2m/w\delta$ which is a finite quantity. Such a finite value would confirm the hypothesis of *superluminality*. However, the *cut off* at $k \approx (1 - \delta)w$ increases the amplitude of the tail of the incident wave as we can observe in Fig. 4. It contributes so relevantly as the peak of the incident wave to the final composition of the transmitted wave and creates an ambiguity in the definition of the *arrival* time.

IV. ONE DIMENSIONAL POTENTIAL SYMMETRICAL SCATTERING FOR $E < V_0$

In order to recover the scattered momentum distribution symmetry conditions for accurately applying the SPM, we assume a symmetrical colliding configuration of two wave packets traveling in opposite directions. By considering the same barrier represented in (40), we solve the Schroedinger equation for a plane wave component of momentum k for

two identical wave packets symmetrically separated from the origin $x = 0$, at time of collision $t = -(mL)/(2k_0)$ chosen for mathematical convenience, where we assume they perform a totally symmetric simultaneous collision with the potential barrier. The wave packet reaching the left(right)-side of the barrier is represented by

$$\psi^{\mathbf{L}(R)}(x, t) = \int_{-\infty}^{+\infty} dk g(k - k_0) \phi^{\mathbf{L}(R)}(k, x) \exp[-i E t] \quad (56)$$

where we assume the integral can be naturally extended from the interval $[0, w]$ to the interval $[-\infty, +\infty]$ as a first approximation. By assuming that $\phi^{\mathbf{L}(R)}(k, x)$ are Schrödinger equation solutions, at the time $t = -(mL)/(2k_0)$ i. e. when the wave packet peaks simultaneously reach the barrier, we can write

$$\phi^{\mathbf{L}(R)}(k, x) = \begin{cases} \phi_1^{\mathbf{L}(R)}(k, x) = \exp[\pm i k x] + R_B^{\mathbf{L}(R)}(k, L) \exp[\mp i k x] & x < -L/2 (x > L/2), \\ \phi_2^{\mathbf{L}(R)}(k, x) = \alpha_B^{\mathbf{L}(R)}(k) \exp[\mp \rho x] + \beta_B^{\mathbf{L}(R)}(k) \exp[\pm \rho x] & -L/2 < x < L/2, \\ \phi_3^{\mathbf{L}(R)}(k, x) = T_B^{\mathbf{L}(R)}(k, L) \exp[\pm i k x] & x > L/2 (x < -L/2). \end{cases}$$

where the upper(lower) sign is related to the index $L(R)$. By assuming the constraints which require the continuity of $\phi^{\mathbf{L},R}$ and their derivatives at $x = -L/2$ and $x = L/2$, after some mathematical manipulations, we can easily obtain

$$R_B^{\mathbf{L},R}(k, L) = \exp[-i k L] \left\{ \frac{\exp[i \Theta(k, L)] [1 - \exp[2 \rho(k) L]]}{1 - \exp[2 \rho(k) L] \exp[i \Theta(k, L)]} \right\} \quad (57)$$

and

$$T_B^{\mathbf{L},R}(k, L) = \exp[-i k L] \left\{ \frac{\exp[\rho(k) L] [1 - \exp[2 i \Theta(k, L)]]}{1 - \exp[2 \rho(k) L] \exp[i \Theta(k, L)]} \right\}, \quad (58)$$

where $\Theta(k, L)$ is given by the Eq. (43) and $R_B^{\mathbf{L}}(k, L)$ and $T_B^R(k, L)$ as well as $R_B^R(k, L)$ and $T_B^{\mathbf{L}}(k, L)$ are intersecting each other. By analogy with the amplitude addition procedure we have adopted in the multiple peak decomposition scattering [14], such a pictorial configuration obliges us to sum the intersecting amplitude of probabilities before taking their squared modulus in order to obtain

$$R_B^{\mathbf{L},R}(k, L) + T_B^R, \mathbf{L}(k, L) = \exp[-i k L] \left\{ \frac{\exp[\rho(k) L] + \exp[i \Theta(k, L)]}{1 + \exp[\rho(k) L] \exp[i \Theta(k, L)]} \right\} = \exp\{-i[k L - \varphi(k, L)]\} \quad (59)$$

with

$$\varphi(k, L) = -\arctan \left\{ \frac{2 k \rho(k) \sinh[\rho(k) L]}{w^2 + (k^2 - \rho^2(k)) \cosh[\rho(k) L]} \right\}. \quad (60)$$

The important information we get from the relation given by (59) is that, differently from the previous standard tunneling analysis, by adding the intersecting amplitudes at each side

of the barrier, we keep the original momentum distribution undistorted since the modulus $|R_B^{\mathbf{L},R}(k, L) + T_B^{R,\mathbf{L}}(k, L)|$ is unitary. At this point we recover the most fundamental condition for the applicability of the SPM which allows us to accurately find the position of the peak of the reconstructed wave packet composed by reflected and transmitted superposing components.

The phase time interpretation can be, in this case, correctly quantified in terms of the analysis of the *new* phase $\varphi(k, L)$. By applying the stationary phase condition to the recomposed wave packets, the maximal point of the scattered amplitudes $g(k - k_0)|R_B^{\mathbf{L},R}(k, L) + T_B^{R,\mathbf{L}}(k, L)|$ are accurately given by $k_{\max} = k_0$ so that the traversal/reflection time or, more generically, the scattering time, results in

$$t_T^\varphi = \frac{m}{k_0} \left. \frac{d\varphi(k, \alpha_{(\mathbf{L})})}{dk} \right|_{k=k_0} = \frac{2mL}{k_0 \alpha} \frac{w^2 \sinh(\alpha) + \alpha k_0^2}{2k_0^2 - w^2 + w^2 \cosh(\alpha)} \quad (61)$$

with α previously defined. It can be said metaphorically that the identical particles represented by both incident wave packets spend a time of the order of t_T^φ inside the barrier before retracing its steps or tunneling. In fact, we cannot differentiate the tunneling from the reflecting waves for such a scattering configuration. The point is that we have introduced a possibility of improving the efficiency of the SPM in calculating reflecting and tunneling phase times by studying a process where the conditions for applying the method are totally recovered, i. e. we have demonstrated that the transmitted and reflected interfering amplitudes results in a unimodular function which just modifies the *envelop* function $g(k - k_0)$ by an additional phase. The previously observed incongruities which cause the distortion of the momentum distribution $g(k - k_0)$ are completely eliminated in this case. At the same time, one could argue about the possibility of extending such a result to the standardly established tunneling process. We should assume that in the region inside the potential barrier, the reflecting and transmitting amplitudes should be summed before we compute the phase changes. Obviously, it would result in the same phase time expression represented by (61). In this case, the assumption of there (not) existing interference between the momentum amplitudes of the reflected and transmitted at the discontinuity points $x = -L/2$ and $x = L/2$ is purely arbitrary. Consequently, it is important to reinforce the argument that such a possibility of interference leading to different phase time results is strictly related to the idea of using (or not) the multiple peak (de)composition in the region where the potential barrier is localized.

In order to illustrate the difference between the standard *tunneling* phase time t_T and the alternative *scattering* phase time t_T^φ we introduce the new parameter $n = k_{\max}^2/w^2$ and the *classical* traversal time $\tau = (mL)/k_{\max}$ in order to define the rates

$$R_T(\alpha) = \frac{t_T}{\tau} = \frac{2}{\alpha} \left\{ \frac{\cosh(\alpha) \sinh(\alpha) - \alpha n (2n-1)}{[4n(1-n) + \sinh^2(\alpha)]} \right\} \quad (62)$$

and

$$R_T^\varphi(\alpha) = \frac{t_T^\varphi}{\tau} = \frac{2}{\alpha} \left\{ \frac{n \alpha + \sinh(\alpha)}{2n-1 + \cosh(\alpha)} \right\} \quad (63)$$

which are plotted in the Fig.(5) for some discrete values of n varying from 0.1 to 0.9, from which we can obtain the most common limits given by

$$\lim_{\alpha \rightarrow \infty} \{R_T^\varphi(\alpha)\} = \lim_{\alpha \rightarrow \infty} \{R_T(\alpha)\} = 0 \quad (64)$$

and

$$\lim_{\alpha \rightarrow 0} \{R_T(\alpha)\} = 1 + \frac{1}{2n}, \quad \lim_{\alpha \rightarrow 0} \{R_T^\varphi(\alpha)\} = 1 + \frac{1}{n} \quad (65)$$

Both of them present the same asymptotic behavior which, at first glance, recover the theoretical possibility of a *superluminal* transmission in the sense that, by now, the SPM can be correctly applied since the analytical limitations are accurately observed. At this point, it is convenient to notice that the superluminal phenomena, observed in the experiments with tunneling photons and evanescent electromagnetic waves [5, 36, 37, 38], has generated a lot of discussions on relativistic causality. In fact, superluminal group velocities in connection with quantum (and classical) tunnelings were predicted even on the basis of tunneling time definitions more general than the simple Wigner's phase-time [34] (Olkhevsky *et al.*, for instance, discuss a simple way of understanding the problem [25]). In a *causal* manner, it might consist in explaining the superluminal phenomena during tunneling as simply due to a *reshaping* of the pulse, with attenuation, as already attempted (at the classical limit) [39], i. e. the later parts of an incoming pulse are preferentially attenuated, in such a way that the outcoming peak appears shifted towards earlier times even if it is nothing but a portion of the incident pulse forward tail [5, 40]. In particular, we do not intend to extend on the delicate question whether superluminal group-velocities can sometimes imply superluminal signalling, a controversial subject which has been extensively explored in the literature about the tunneling effect ([25] and references therein).

Turning back to the scattering time analysis, we can observe an analogy between our results and the results interpreted from the Hartman Effect (HE) analysis [8]. The HE is related to the fact that for opaque potential barriers the mean tunneling time does not depend

on the barrier width, so that for large barriers the effective tunneling-velocity can become arbitrarily large, where it was found that the tunneling phase-time was independent of the barrier width. It seems that the penetration time, needed to cross a portion of a barrier, in the case of a very long barrier starts to increase again after the plateau corresponding to infinite speed proportionally to the distance [41]. Our phase time dependence on the barrier width is similar to that which leads to Hartman interpretation as we can infer from Eqs. (64-65). Only when α tends to 0 we have an explicit linear time-dependence on L given by

$$t_T^\varphi = \frac{2mL}{w} \left(1 + \frac{1}{n} \right) \quad (66)$$

which agree with calculations based on the simple phase-time analysis where $t_T = \frac{2mL}{w} \left(1 + \frac{1}{2n} \right)$. However, it is important to emphasize that the wave packets for which we compute the phase times illustrated in the Fig.(5) are not *effectively* constructed with the same momentum distributions. The phase $\Theta(k, L)$ appears when we treat separately each momentum amplitude $g(k - k_0) |T(k, L)|$ and $g(k - k_0) |R(k, L)|$ and the other one $\varphi(k, L)$ appears when we sum the amplitudes on the same side of the barrier. The summed amplitude corresponds to a recomposed symmetrical momentum distribution for which the SPM will give the time-position of the peak of the corresponding wave packet. In this sense, for overcoming the abovementioned incongruities, we have reconsidered the use of the multiple peak decomposition technique previously presented in the study of the above barrier diffusion problem [14]

V. CONCLUSION

In this manuscript we have discussed the wave packet multiple peak decomposition in order to identify some incongruities due to erroneous application of the method of stationary phase for deriving phase times related to wave packets scattered by a unidimensional potential barrier. The main point on which we have elaborated shows us that the results of the SPM depend critically upon the manipulation of the amplitude prior to the application of the method. The method is inherently ambiguous unless we know, by some other means, at least the number of separate peaks involved. By analyzing the above barrier diffusion problem, we have demonstrated that the barrier results can be obtained by treating the

barrier as a two-step process. This procedure involves multiple reflections at each step and predict the existence of multiple (infinite) outgoing peaks.

In the same analysis which generically involves analytically-continuous *gaussian* pulses, holomorphic functions which do not have a well-defined front, we have quantified some aspects inherent to the tunneling phenomenon. We have observed that the amplitude of the transmitted wave is essentially composed by the plane wave components of the front tail of the *incoming* wave packet which reaches the first barrier interface before the peak arrival. As we have noticed, this *filter effect* by itself leads to ambiguities in the interpretation of the wave packet speed of propagation. To overcome these incompatibilities which leads to the misconception of the tunneling phase time, we have considered the multiple peak decomposition technique in a suitable way for comprehending the conservation of probabilities for a very particular tunneling configuration. An example for which, we believe, we have provided a simple but convincing resolution, since the asymmetry presented in the previous case was eliminated, and the phase times could be accurately computed.

We let for a subsequent analysis the suggestive possibility of investigating the validity of our approach when confronting with the intriguing case of multiple opaque barriers [35], in particular, in the case of non-resonant tunneling. Still concerning with the future theoretical perspectives, the symmetrical colliding configuration also offers the possibility of exploring some applications involving soliton structures. In summary, the above discussion reinforces the assertion that the investigation of wave propagation across a tunnel barrier has always been an intriguing subject which is wide open both from a theoretical and an experimental point of view.

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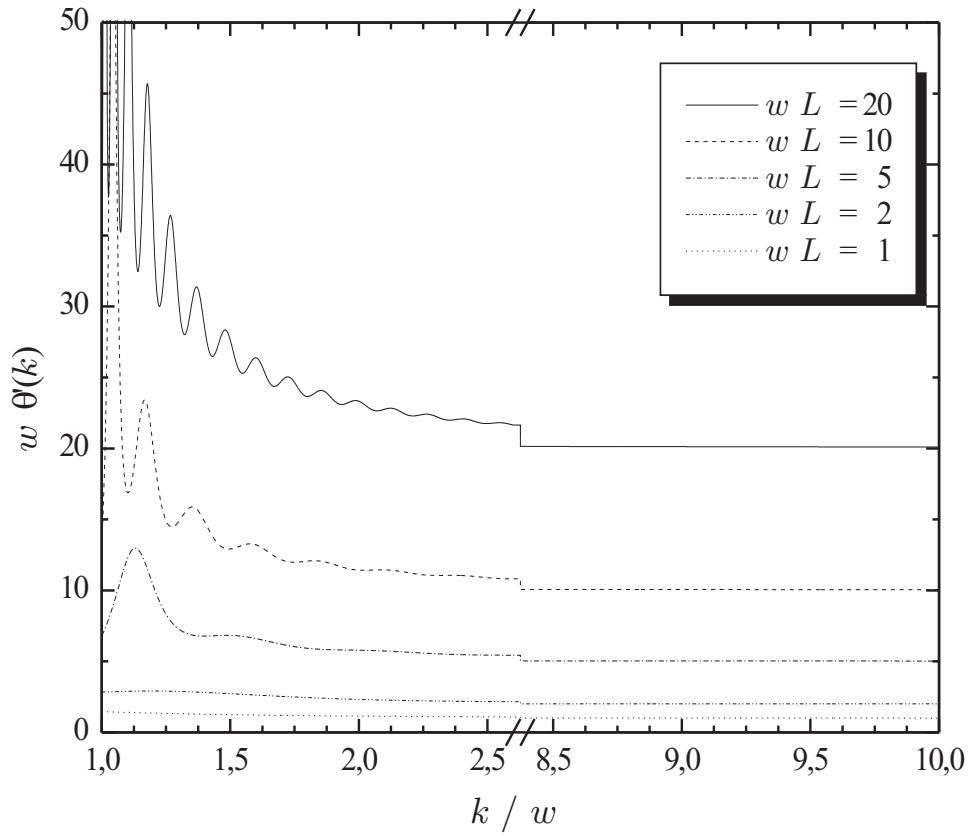


FIG. 1: The phase derivative $\Theta'(k)$ dependence on k/w . $\Theta'(k)$ does not present an adequate analytical behavior (smoothness) for the applicability of the SPM when k approximates to w since the phase derivative oscillates too rapidly (The phase is not stationary). The method can be accurately applied for larger values of k/w when the phase is really stationary.

TABLE I: The values of k numerically obtained by increasing the barrier extension L . The values are calculated in terms of the wave packet width a for different values of the potential barrier high expressed in terms of wa . We have fixed the incoming momentum by setting $k_0 a$ equal to 1.

wa	1.5	2.0	4.0	6.0	8.0	10	20
L/a							
0.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0062	1.0188	1.1777	1.4156	1.6238	1.7726	1.9834
0.10	1.0235	1.0648	1.3799	1.6769	1.8547	1.9397	2.0051
0.15	1.0489	1.1223	1.5349	1.8251	1.9505	1.9937	2.0133
0.20	1.0794	1.1825	1.6571	1.9178	2.0000	2.0204	2.0203
0.25	1.1129	1.2420	1.7575	1.9813	2.0317	2.0390	2.0272
0.30	1.1478	1.3001	1.8430	2.0289	2.0562	2.0551	2.0342
0.35	1.1836	1.3565	1.9185	2.0679	2.0779	2.0704	2.0413
0.40	1.2196	1.4116	1.9874	2.1025	2.0986	2.0857	2.0484
0.45	1.2558	1.4657	2.0524	2.1350	2.1191	2.1012	2.0556
0.50	1.2921	1.5194	2.1155	2.1668	2.1399	2.1170	2.0628
0.55	1.3285	1.5729	2.1785	2.1988	2.1611	2.1331	2.0701
0.60	1.3649	1.6266	2.2429	2.2314	2.1828	2.1495	2.0775
0.65	1.4015	1.6809	2.3101	2.2651	2.2051	2.1663	2.0850
0.70	1.4383	1.7360	2.3819	2.3002	2.2281	2.1834	2.0925
0.75	1.4751	1.7920	2.4599	2.3367	2.2518	2.2009	2.1001
0.80	*	1.8489	2.5466	2.3751	2.2761	2.2188	2.1078
0.85	*	1.9065	2.6456	2.4154	2.3013	2.2371	2.1155
0.90	*	1.9646	2.7627	2.4578	2.3272	2.2558	2.1234
0.95	*	*	2.9091	2.5028	2.3540	2.2750	2.1313
1.00	*	*	3.1137	2.5504	2.3818	2.2947	2.1392

^a For the values of L marked with *, we can demonstrate by means of Eqs. (54-55) that the modulated momentum distribution has already been completely distorted and the maximum loses its meaning in the context of applicability of the method of stationary phase.

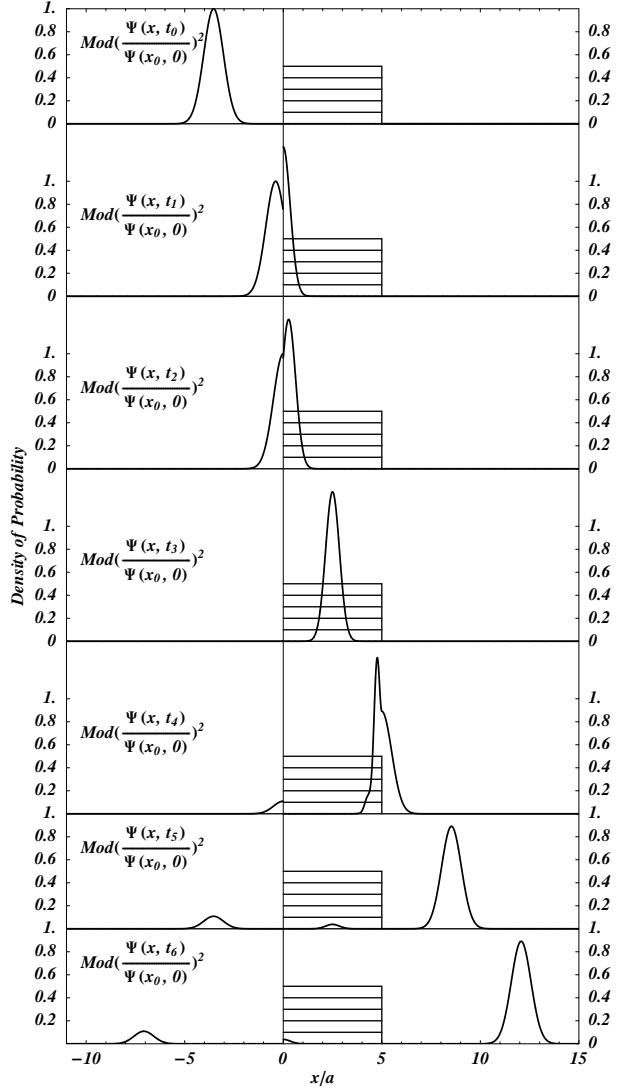


FIG. 2: Erroneous interpretation of the scattering of an incoming wave packet by a unidimensional potential barrier of height V_0 and width L . It has the purpose of illustrating the deficiencies inherent to the *wrong* applicability of the SPM. We have plotted the propagating wave packets at the corresponding times $t_n = (ma^2)[n(L/a)]/(a q_0)$ (with $n = 0, 1, \dots, 5$ and with the normalization constraint $ma^2 = 1$) by assuming the incoming wave packet starts at $x = -(k_0 L)/(2q_0)$. From the *false* behavior of the density of probabilities it becomes obvious that the total probability is not conserved as it was expected. The square of the amplitude modulus would supposedly represent a collision of a wave packet of average width a with a potential barrier V_0 of width $L = 5a$ where, for illustrating reasons, we have adopted $k_0 = \sqrt{2}w$ and $wa = 10000$. Only a fixed region in x close to the barrier is shown.

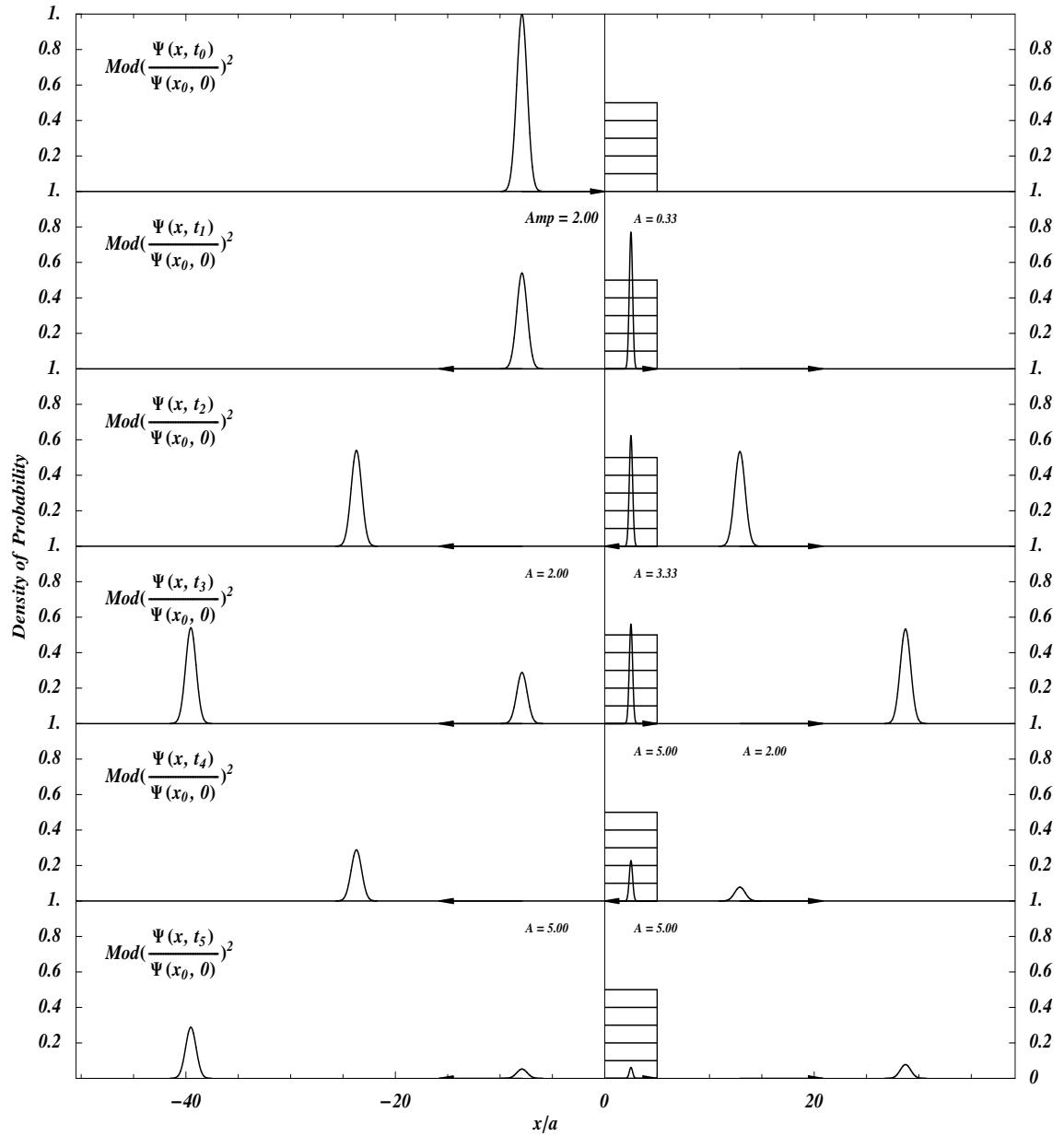


FIG. 3: Multiple peak decomposition of a propagating wave packet in the above potential barrier scattering problem. We have plotted the first few reflected and transmitted waves for times $t_n = (ma^2)[n(L/a)]/(a q_0)$ (with $n = 0, 1, \dots, 5$ and $ma^2 = 1$) in correspondence with Eq. (34), by assuming that the incoming wave packet starts at $x = -(k_0 L)/(2q_0)$. The density of probabilities represents the collision of a wave packet of average width a with a potential barrier V_0 of width $L = 5a$. Just for illustrating reasons, we have adopted $k_0 = (\sqrt{10}w)/3$ ($wa = 10000$) and we have printed the wave packet amplification multiplying factor (A) (individually adopted for visual convenience for each wave packet) when necessary.

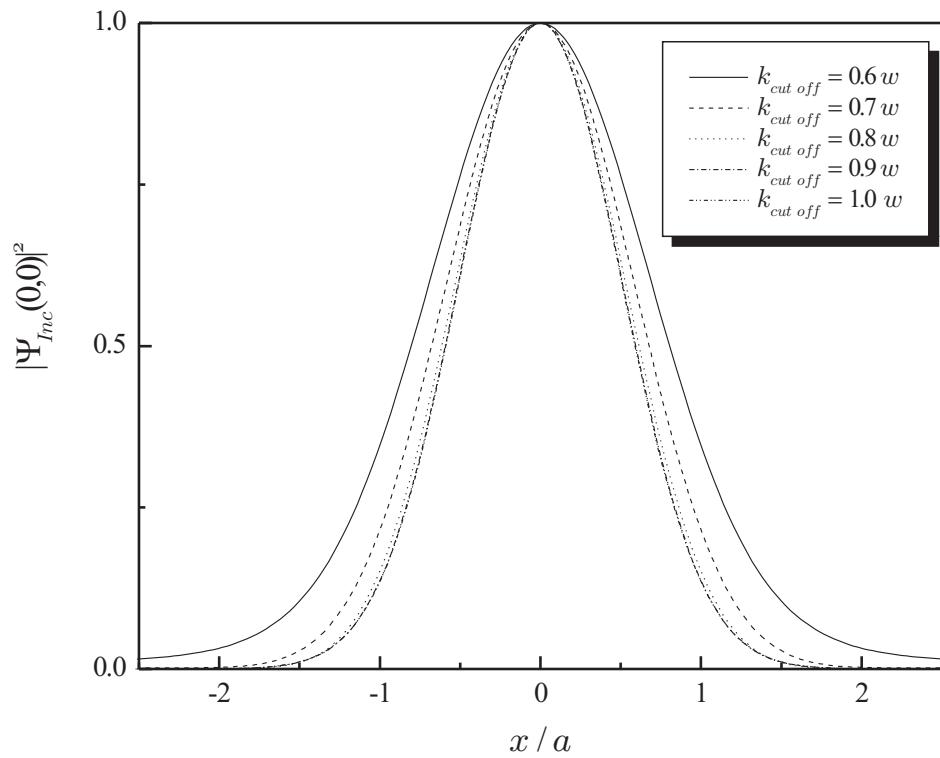


FIG. 4: Dependence of the wave packet shape on the *cut off* value of a momentum distribution centered around $k_o = 0.5w$ with the values of k comprised between 0 and $k_{cut off}$.

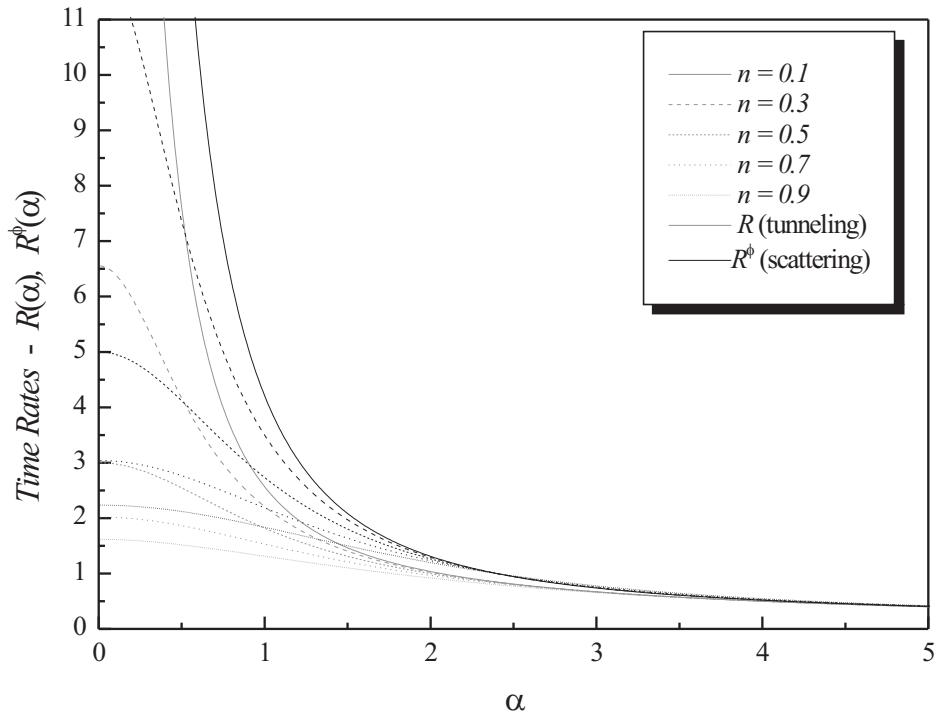


FIG. 5: Time rates for the *standard* tunneling and the *new* scattering process. The rates $R(\alpha)$ and $R^\phi(\alpha)$ can be understood as transmitted times in the unities of the classical propagation time τ . Both of them present the same asymptotic behavior.